The Interiors of Stars

PROBLEM SET

1. Show that the equation for hydrostatic equilibrium, Eq. (6), can also be written in terms of the optical depth \( \tau \), as

\[
\frac{dP}{d\tau} = \frac{g}{\kappa}.
\]

This form of the equation is often useful in building model stellar atmospheres.

\[
\frac{dP}{d\tau} = -G \frac{M \rho}{r^2} = -\rho g,
\] (6)

2. Prove that the gravitational force on a point mass located anywhere inside a hollow, spherically symmetric shell is zero. Assume that the mass of the shell is \( M \) and has a constant density \( \rho \). Assume also that the radius of the inside surface of the shell is \( r_1 \) and that the radius of the outside surface is \( r_2 \). The mass of the point is \( m \).

3. Assuming that 10 eV could be released by every atom in the Sun through chemical reactions, estimate how long the Sun could shine at its current rate through chemical processes alone. For simplicity, assume that the Sun is composed entirely of hydrogen. Is it possible that the Sun’s energy is entirely chemical? Why or why not?

4. (a) Taking into consideration the Maxwell–Boltzmann velocity distribution, what temperature would be required for two protons to collide if quantum mechanical tunneling is neglected? Assume that nuclei having velocities ten times the root-mean-square (rms) value for the Maxwell–Boltzmann distribution can overcome the Coulomb barrier. Compare your answer with the estimated central temperature of the Sun.

(b) Using the below equation, calculate the ratio of the number of protons having velocities ten times the rms value to those moving at the rms velocity.

\[
n_v dv = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} \frac{4\pi v^2}{v} dv,
\]

(c) Assuming (incorrectly) that the Sun is pure hydrogen, estimate the number of hydrogen nuclei in the Sun. Could there be enough protons moving with a speed ten times the rms value to account for the Sun’s luminosity?

5. Derive the ideal gas law, Eq. (10). Begin with the pressure integral (Eq. 9) and the Maxwell–Boltzmann velocity distribution function,

\[
n_v dv = n \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT} \frac{4\pi v^2}{v} dv,
\]

\[
P_v = nkT
\]

\[
P = \frac{1}{3} \int_0^\infty mn_v v^4 dv,
\] (9)
The Interiors of Stars: Problem Set

6 Derive Eq. (28) from the following equation:

\[ n_v \, d v = n \left( \frac{m}{2 \pi k T} \right)^{3/2} e^{-m v^2 / 2kT} 4\pi v^2 \, dv. \]

\[ n_E \, d E = \frac{2n}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} \, dE \]  

(28)

7 By invoking the virial theorem (see below), make a crude estimate of an “average” temperature for the Sun. Is your result consistent with other estimates obtained in “The Interiors of Stars”? Why or why not?

\[ -2 \langle K \rangle = \langle U \rangle. \]

8 Show that the form of the Coulomb potential barrier penetration probability given by Eq. (31) follows directly from Eq. (30).

\[ \sigma(E) \propto e^{-2\pi^2 U_c / E}. \]  

(30)

\[ \sigma(E) \propto e^{-bE^{-1/2}}. \]  

(31)

9 Prove that the energy corresponding to the Gamow peak is given by Eq. (34).

\[ E_0 = \left( \frac{b k T}{2} \right)^{2/3}. \]  

(34)

10 Calculate the ratio of the energy generation rate for the pp chain to the energy generation rate for the CNO cycle given conditions characteristic of the center of the present-day (evolved) Sun, namely \( T = 1.5696 \times 10^7 \) K, \( \rho = 1.527 \times 10^5 \) kg m\(^{-3}\), \( X = 0.3397 \), and \( X_{\text{CNO}} = 0.0141 \). Assume that the pp chain screening factor is unity \( (f_{pp} = 1) \) and that the pp chain branching factor is unity \( (\psi_{pp} = 1) \).

11 Beginning with Eq. (62) and writing the energy generation rate in the form

\[ \epsilon(T) = \epsilon^* T_8^4, \]

show that the temperature dependence for the triple alpha process, given by Eq. (63), is correct. \( \epsilon^* \) is a function that is independent of temperature.

Hint: First take the natural logarithm of both sides of Eq. (62) and then differentiate with respect to \( \ln T_8 \). Follow the same procedure with your power law form of the equation and compare the results. You may want to make use of the relation

\[ \frac{d \ln \epsilon}{d \ln T_8} = \frac{d \ln \epsilon}{d T_8} \frac{1}{T_8} d T_8 = T_8 \frac{d \ln \epsilon}{d T_8}. \]

\[ \epsilon_{3\alpha} = 50.9 \rho^2 Y^3 T_8^{-3} f_{3\alpha} e^{-44.027 T_8^{-1}} \, \text{W kg}^{-1}. \]  

(62)

\[ \epsilon_{3\alpha} \simeq \epsilon_{3\alpha}' \rho^2 Y^3 f_{3\alpha} T_8^{4.10}. \]  

(63)

The Interiors of Stars: Problem Set

12 The $Q$ value of a reaction is the amount of energy released (or absorbed) during the reaction. Calculate the $Q$ value for each step of the PP I reaction chain (Eqs. 37–39). Express your answers in MeV. The masses of $^2\text{H}$ and $^3\text{He}$ are 2.0141 u and 3.0160 u, respectively.

$$^1\text{H} + ^1\text{H} \rightarrow ^2\text{H} + e^+ + \nu_e \quad (37)$$

$$^2\text{H} + ^1\text{H} \rightarrow ^3\text{He} + \gamma \quad (38)$$

$$^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2 ^1\text{H}. \quad (39)$$

13 Calculate the amount of energy released or absorbed in the following reactions (express your answers in MeV):

(a) $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{24}\text{Mg} + \gamma$

(b) $^{12}\text{C} + ^{12}\text{C} \rightarrow ^{16}\text{O} + 2 ^{3}\text{He}$

(c) $^{19}\text{F} + ^1\text{H} \rightarrow ^{16}\text{O} + ^{3}\text{He}$

The mass of $^{12}\text{C}$ is 12.0000 u, by definition, and the masses of $^{16}\text{O}$, $^{19}\text{F}$, and $^{24}\text{Mg}$ are 15.99491 u, 18.99840 u, and 23.98504 u, respectively. Are these reactions exothermic or endothermic?

14 Complete the following reaction sequences. Be sure to include any necessary leptons.

(a) $^{27}\text{Si} \rightarrow ^{27}\text{Al} + e^+ + \gamma$

(b) $^{7}\text{Li} + ^1\text{H} \rightarrow ^{24}\text{Mg} + ^1\text{H}$

(c) $^{35}\text{Cl} + ^1\text{H} \rightarrow ^{36}\text{Ar} + ^1\text{H}$

15 Prove that Eq. (83) follows from Eq. (82).

$$PV = K, \quad (82)$$

$$P = K'T^{\gamma/(\gamma-1)}, \quad (83)$$

16 Show that Eq. (109) can be obtained from Eq. (108).

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dP}{dr} \right) = -4\pi G\rho. \quad (108)$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G\rho, \quad (109)$$

17 Starting with the Lane–Emden equation and imposing the necessary boundary conditions, prove that the $n = 0$ polytrope has a solution given by

$$D_0(\xi) = 1 - \frac{\xi^2}{6}, \quad \text{with} \; \xi_1 = \sqrt{6}.$$

18 Derive an expression for the total mass of an $n = 5$ polytrope, and show that although $\xi_1 \rightarrow \infty$, the mass is finite.

19 On the same graph, plot the density structure of stars of polytropic indices $n = 0$, $n = 1$, and $n = 5$. Hint: You will want to plot $\rho_0/\rho$, vs. $r/r_m$.

(b) What can you conclude about the concentration of density with radius for increasing polytropic index?

(c) From the trend that you observe for the analytic solutions to the Lane–Emden equation, what would you expect regarding the density concentration of an adiabatically convective stellar model compared to a model in radiative equilibrium?

(d) Explain your conclusion in part (c) in terms of the physical processes of convection and radiation.
Estimate the hydrogen-burning lifetimes of stars near the lower and upper ends of the main sequence. The lower end of the main sequence occurs near 0.072 M⊙, with log₁₀(Tₑ) = 3.23 and log₁₀(L/L⊙) = −4.3. On the other hand, an 85 M⊙ star near the upper end of the main sequence has an effective temperature and luminosity of log₁₀(Tₑ) = 4.705 and log₁₀(L/L⊙) = 6.006, respectively. Assume that the 0.072 M⊙ star is entirely convective so that, through convective mixing, all of its hydrogen, rather than just the inner 10%, becomes available for burning.

Using the information given in Problem 21 above, calculate the radii of a 0.072 M⊙ star and an 85 M⊙ star. What is the ratio of their radii?

(a) Estimate the Eddington luminosity of a 0.072 M⊙ star and compare your answer to the main-sequence luminosity given in Problem 21. Assume σ = 0.001 m² kg⁻¹. Is radiation pressure likely to be significant in the stability of a low-mass main-sequence star?

(b) If a 120 M⊙ star forms with log₁₀(Tₑ) = 4.727 and log₁₀(L/L⊙) = 6.252, estimate its Eddington luminosity. Compare your answer with the actual luminosity of the star.

**COMPUTER PROBLEMS**

(a) Use a numerical integration algorithm such as a Runge–Kutta method to compute the density profile for the n = 1.5 and n = 3 polytropes. Be sure to correctly incorporate the boundary conditions in your integrations.

(b) Plot your results and compare them with the n = 0, n = 1, and n = 5 analytic models determined in Problem 20.

Verify that the basic equations of stellar structure [Eqs. (6), (7), (36), (68)] are satisfied by the 1 M⊙ StatStar model available for download from the companion website; see Appendix: StatStar, A Stellar Structure Code. This may be done by selecting two adjacent zones and numerically computing the derivatives on the left-hand sides of the equations, for example

\[
\frac{dP}{dr} \approx \frac{P_{i+1} - P_{i}}{r_{i+1} - r_{i}},
\]

and comparing your results with results obtained from the right-hand sides using average values of quantities for the two zones [e.g., \( M_r = (M_i + M_{i+1})/2 \)].

Perform your calculations for two adjacent shells at temperatures near 5 × 10⁶ K, and then compare your results for the left- and right-hand sides of each equation by determining relative errors. Note that the model assumes complete ionization everywhere and has the uniform composition \( X = 0.7, Y = 0.292, Z = 0.008 \). Your results on the left- and right-hand sides of the stellar structure equations will not agree exactly because StatStar uses a Runge–Kutta numerical algorithm that carries out intermediate steps not shown in the output file.

\[
\frac{dP}{dr} = -G \frac{M_r \rho}{r^2} = -\rho g, \tag{6}
\]

\[
\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{7}
\]


The Interiors of Stars: Problem Set

\[ \frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon, \quad (36) \]

\[ \frac{dT}{dr} = -\frac{3}{4ac} \left( \frac{\kappa \rho}{T^3} \right) L_r, \quad (68) \]

26 The companion website contains an example of a theoretical 1.0 M\(_{\odot}\) main-sequence star produced by the stellar structure code StatStar, described in Appendix: StatStar, A Stellar Structure Code. Using StatStar, build a second main-sequence star with a mass of 0.75 M\(_{\odot}\) that has a homogeneous composition of \( X = 0.7, \ Y = 0.292, \) and \( Z = 0.008. \) For these values, the model’s luminosity and effective temperature are 0.189 L\(_{\odot}\) and 3788.5 K, respectively. Compare the central temperatures, pressures, densities, and energy generation rates between the 1.0 M\(_{\odot}\) and 0.75 M\(_{\odot}\) models. Explain the differences in the central conditions of the two models.

27 Use the stellar structure code StatStar described in Appendix: StatStar, A Stellar Structure Code, together with the theoretical StatStar H–R diagram and mass–effective temperature data provided on the companion website, to calculate a homogeneous, main-sequence model having the composition \( X = 0.7, \ Y = 0.292, \) and \( Z = 0.008. \) (Note: It may be more illustrative to assign each student in the class a different mass for this problem so that the results can be compared.)

(a) After obtaining a satisfactory model, plot \( P \) versus \( r, \ M_r \) versus \( r, \ L_r \) versus \( r, \) and \( T \) versus \( r. \)

(b) At what temperature has \( L_r \) reached approximately 99% of its surface value? 50% of its surface value? Is the temperature associated with 50% of the total luminosity consistent with the rough estimate found in Eq. (27)? Why or why not?

\[ T_{\text{quantum}} = \frac{Z_1^2 Z_2^2 e^\mu_m}{12\pi^2 \hbar^2 k}. \quad (27) \]

(c) What are the values of \( M_r / M_\star \) for the two temperatures found in part (b)? \( M_\star \) is the total mass of the stellar model.

(d) If each student in the class calculated a different mass, compare the changes in the following quantities with mass:

(i) The central temperature.

(ii) The central density.

(iii) The central energy generation rate.

(iv) The extent of the central convection zone with mass fraction and radius.

(v) The effective temperature.

(vi) The radius of the star.

(e) If each student in the class calculated a different mass:

(i) Plot each model on a graph of luminosity versus mass (i.e., plot \( L_\star / L_\odot \) versus \( M_\star / M_\odot \)).

(ii) Plot \( \log_{10}(L_\star / L_\odot) \) versus \( \log_{10}(M_\star / M_\odot) \) for each stellar model.

(iii) Using an approximate power law relation of the form

\[ L_\star / L_\odot = (M_\star / M_\odot)^\alpha, \]

find an appropriate value for \( \alpha. \) \( \alpha \) may differ for different compositions or vary somewhat with mass. This is known as the mass–luminosity relation (see below figure).