GLOBAL EDITION

Calculus

for Business, Economics, Life Sciences, and Social Sciences

FOURTEENTH EDITION

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fourteenth edition

CALCULUS

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Pearson

(B) Find the supply and the instantaneous rate of change of supply with respect to price when the price is \$40. Write a brief verbal interpretation of these results.

(C) Use the results from part (B) to estimate the supply if the price is increased to \$41.

97. Medicine. A drug is injected into a patient's bloodstream through her right arm. The drug concentration (in milligrams per cubic centimeter) in the bloodstream of the left arm t hours after the injection is given by

$$C(t) = \frac{0.14t}{t^2 + 1}$$

(A) Find $C'(t)$.

(B) Find $C'(0.5)$ and $C'(3)$, and interpret the results.

98. Drug sensitivity. One hour after a dose of x milligrams of a particular drug is administered to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by

$$T(x) = x^2\left(1 - \frac{x}{9}\right) \qquad 0 \le x \le 7$$

The rate $T'(x)$ at which T changes with respect to the size of the dosage x is called the *sensitivity* of the body to the dosage.

(A) Use the product rule to find $T'(x)$.

(B) Find $T'(1)$, $T'(3)$, and $T'(6)$.

Answers to Matched Problems

1. $30x^4 - 36x^3 + 9x^2$

2. (A) $y = 84x - 297$

(B) $x = -4, x = 1$

3. (A) $5x^8e^x + e^x(40x^7) = 5x^7(x + 8)e^x$

(B) $x^7 \cdot \frac{1}{x} + \ln x\,(7x^6) = x^6\,(1 + 7\ln x)$

4. (A) $\dfrac{(x^2 + 3)2 - (2x)(2x)}{(x^2 + 3)^2} = \dfrac{6 - 2x^2}{(x^2 + 3)^2}$

(B) $\dfrac{(t^2 - 4)(3t^2 - 3) - (t^3 - 3t)(2t)}{(t^2 - 4)^2} = \dfrac{t^4 - 9t^2 + 12}{(t^2 - 4)^2}$

(C) $-\dfrac{6}{x^4}$

5. (A) $\dfrac{(e^x + 2)\,3x^2 - x^3e^x}{(e^x + 2)^2}$

(B) $\dfrac{(1 + \ln x)\,4 - 4x\frac{1}{x}}{(1 + \ln x)^2} = \dfrac{4\ln x}{(1 + \ln x)^2}$

6. (A) $S'(t) = \dfrac{450}{(t + 3)^2}$

(B) $S(12) = 120$; $S'(12) = 2$. After 12 months, the total sales are 120,000 games, and sales are increasing at the rate of 2,000 games per month.

(C) 122,000 games

3.5 The Chain Rule

- Composite Functions
- General Power Rule
- The Chain Rule

The word *chain* in the name "chain rule" comes from the fact that a function formed by composition involves a chain of functions—that is, a function of a function. The *chain rule* enables us to compute the derivative of a composite function in terms of the derivatives of the functions making up the composite. In this section, we review composite functions, introduce the chain rule by means of a special case known as the *general power rule,* and then discuss the chain rule itself.

Composite Functions

The function $m(x) = (x^2 + 4)^3$ is a combination of a quadratic function and a cubic function. To see this more clearly, let

$$y = f(u) = u^3 \qquad \text{and} \qquad u = g(x) = x^2 + 4$$

We can express y as a function of x:

$$y = f(u) = f[g(x)] = [x^2 + 4]^3 = m(x)$$

The function m is the *composite* of the two functions f and g.

DEFINITION Composite Functions

A function m is a **composite** of functions f and g if

$$m(x) = f[g(x)]$$

The domain of m is the set of all numbers x such that x is in the domain of g, and $g(x)$ is in the domain of f.

The composite m of functions f and g is pictured in Figure 1. The domain of m is the shaded subset of the domain of g (Fig. 1); it consists of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f. Note that the functions f and g play different roles. The function g, which is on the *inside* or *interior* of the square brackets in $f[g(x)]$, is applied first to x. Then function f, which appears on the *outside* or *exterior* of the square brackets, is applied to $g(x)$, provided $g(x)$ is in the domain of f. Because f and g play different roles, the composite of f and g is usually a different function than the composite of g and f, as illustrated by Example 1.

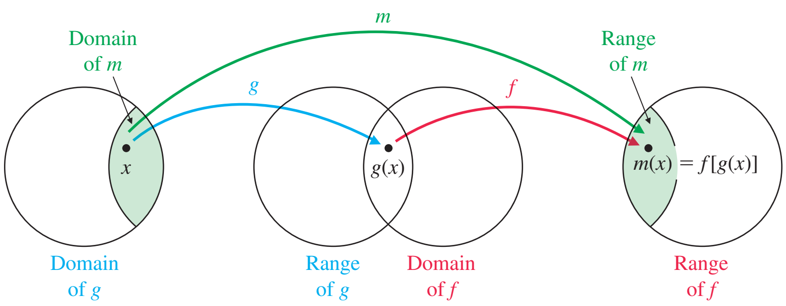

Figure 1 **The composite m of f and g**

EXAMPLE 1 **Composite Functions** Let $f(u) = e^u$ and $g(x) = -3x$. Find $f[g(x)]$ and $g[f(u)]$.

SOLUTION

$$f[g(x)] = f(-3x) = e^{-3x}$$
$$g[f(u)] = g(e^u) = -3e^u$$

Matched Problem 1 Let $f(u) = 2u$ and $g(x) = e^x$. Find $f[g(x)]$ and $g[f(u)]$.

EXAMPLE 2 **Composite Functions** Write each function as a composite of two simpler functions.

(A) $y = 100e^{0.04x}$ (B) $y = \sqrt{4 - x^2}$

SOLUTION

(A) Let

$$y = f(u) = 100e^u$$
$$u = g(x) = 0.04x$$

Check: $y = f[g(x)] = f(0.04x) = 100e^{0.04x}$

(B) Let

$$y = f(u) = \sqrt{u}$$
$$u = g(x) = 4 - x^2$$

Check: $y = f[g(x)] = f(4 - x^2) = \sqrt{4 - x^2}$

Matched Problem 2 Write each function as a composite of two simpler functions.

(A) $y = 50e^{-2x}$ (B) $y = \sqrt[3]{1 + x^3}$

CONCEPTUAL INSIGHT

There can be more than one way to express a function as a composite of simpler functions. Choosing $y = f(u) = 100u$ and $u = g(x) = e^{0.04x}$ in Example 2A produces the same result:

$$y = f[g(x)] = 100g(x) = 100e^{0.04x}$$

Since we will be using composition as a means to an end (finding a derivative), usually it will not matter which functions you choose for the composition.

General Power Rule

We have already made extensive use of the power rule,

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1}$$

Can we apply rule (1) to find the derivative of the composite function $m(x) = p[u(x)] = [u(x)]^n$, where p is the power function $p(u) = u^n$ and $u(x)$ is a differentiable function? In other words, is rule (1) valid if x is replaced by $u(x)$?

Explore and Discuss 1

Let $u(x) = 2x^2$ and $m(x) = [u(x)]^3 = 8x^6$. Which of the following is $m'(x)$?

(A) $3[u(x)]^2$ (B) $3[u'(x)]^2$ (C) $3[u(x)]^2u'(x)$

The calculations in Explore and Discuss 1 show that we cannot find the derivative of $[u(x)]^n$ simply by replacing x with $u(x)$ in equation (1).

How can we find a formula for the derivative of $[u(x)]^n$, where $u(x)$ is an arbitrary differentiable function? Let's begin by considering the derivatives of $[u(x)]^2$ and $[u(x)]^3$ to see if a general pattern emerges. Since $[u(x)]^2 = u(x)u(x)$, we use the product rule to write

$$\begin{aligned}\frac{d}{dx}[u(x)]^2 &= \frac{d}{dx}[u(x)u(x)] \\ &= u(x)u'(x) + u(x)u'(x) \\ &= 2u(x)u'(x)\end{aligned} \tag{2}$$

Because $[u(x)]^3 = [u(x)]^2u(x)$, we use the product rule and the result in equation (2) to write

$$\begin{aligned}\frac{d}{dx}[u(x)]^3 &= \frac{d}{dx}\{[u(x)]^2u(x)\} \\ &= [u(x)]^2\frac{d}{dx}u(x) + u(x)\frac{d}{dx}[u(x)]^2 \\ &= [u(x)]^2u'(x) + u(x)[2u(x)u'(x)] \\ &= 3[u(x)]^2u'(x)\end{aligned}$$

Use equation (2) to substitute for $\frac{d}{dx}[u(x)]^2$.

Continuing in this fashion, we can show that

$$\frac{d}{dx}[u(x)]^n = n[u(x)]^{n-1}u'(x) \qquad n \text{ a positive integer} \tag{3}$$

Using more advanced techniques, we can establish formula (3) for all real numbers n, obtaining the **general power rule**.

THEOREM 1 General Power Rule

If $u(x)$ is a differentiable function, n is any real number, and

$$y = f(x) = [u(x)]^n$$

then

$$f'(x) = n[u(x)]^{n-1}u'(x)$$

Using simplified notation,

$$y' = nu^{n-1}u' \qquad \text{or} \qquad \frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \text{where } u = u(x)$$

EXAMPLE 3 **Using the General Power Rule** Find the indicated derivatives:

(A) $f'(x)$ if $f(x) = (3x + 1)^4$

(B) y' if $y = (x^3 + 4)^7$

(C) $\dfrac{d}{dt}\dfrac{1}{(t^2 + t + 4)^3}$

(D) $\dfrac{dh}{dw}$ if $h(w) = \sqrt{3 - w}$

SOLUTION

(A)
$$\begin{aligned} f(x) &= (3x + 1)^4 && \text{Apply general power rule.} \\ f'(x) &= 4(3x + 1)^3(3x + 1)' && \text{Substitute } (3x + 1)' = 3. \\ &= 4(3x + 1)^3\,3 && \text{Simplify.} \\ &= 12(3x + 1)^3 \end{aligned}$$

(B)
$$\begin{aligned} y &= (x^3 + 4)^7 && \text{Apply general power rule.} \\ y' &= 7(x^3 + 4)^6(x^3 + 4)' && \text{Substitute } (x^3 + 4)' = 3x^2. \\ &= 7(x^3 + 4)^6\,3x^2 && \text{Simplify.} \\ &= 21x^2(x^3 + 4)^6 \end{aligned}$$

(C) $\dfrac{d}{dt}\dfrac{1}{(t^2+t+4)^3}$

$= \dfrac{d}{dt}(t^2+t+4)^{-3}$ Apply general power rule.

$\boxed{= -3(t^2+t+4)^{-4}(t^2+t+4)'}$ Substitute $(t^2+t+4)' = 2t+1$.

$= -3(t^2+t+4)^{-4}(2t+1)$ Simplify.

$= \dfrac{-3(2t+1)}{(t^2+t+4)^4}$

(D) $h(w) = \sqrt{3-w} = (3-w)^{1/2}$ Apply general power rule.

$\dfrac{dh}{dw} \boxed{= \dfrac{1}{2}(3-w)^{-1/2}(3-w)'}$ Substitute $(3-w)' = -1$.

$= \dfrac{1}{2}(3-w)^{-1/2}(-1)$ Simplify.

$= -\dfrac{1}{2(3-w)^{1/2}}$ or $-\dfrac{1}{2\sqrt{3-w}}$

Matched Problem 3 Find the indicated derivatives:

(A) $h'(x)$ if $h(x) = (5x+2)^3$

(B) y' if $y = (x^4-5)^5$

(C) $\dfrac{d}{dt}\dfrac{1}{(t^2+4)^2}$

(D) $\dfrac{dg}{dw}$ if $g(w) = \sqrt{4-w}$

Notice that we used two steps to differentiate each function in Example 3. First, we applied the general power rule, and then we found du/dx. As you gain experience with the general power rule, you may want to combine these two steps. If you do this, be certain to multiply by du/dx. For example,

$\dfrac{d}{dx}(x^5+1)^4 = 4(x^5+1)^3 5x^4$ Correct

$\dfrac{d}{dx}(x^5+1)^4 \neq 4(x^5+1)^3$ $du/dx = 5x^4$ is missing

CONCEPTUAL INSIGHT

If we let $u(x) = x$, then $du/dx = 1$, and the general power rule reduces to the (ordinary) power rule discussed in Section 2.5. Compare the following:

$\dfrac{d}{dx}x^n = nx^{n-1}$ Yes—power rule

$\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}$ Yes—general power rule

$\dfrac{d}{dx}u^n \neq nu^{n-1}$ Unless $u(x) = x + k$, so that $du/dx = 1$