THOMAS' CALCULUS

Early Transcendentals

Fourteenth Edition in SI Units

Hass • Heil • Weir

Proof of Part (a)

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Additivity Rule for Definite Integrals}\\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Order of Integration Rule}\\
&= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx && \text{Let } u=-x,\ du=-dx. \text{ When } x=0,\ u=0. \text{ When } x=-a,\ u=a.\\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx\\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx && f \text{ is even, so } f(-u)=f(u).\\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

The proof of part (b) is entirely similar and you are asked to give it in Exercise 116. ■

EXAMPLE 3 Evaluate $\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\begin{aligned}
\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx\\
&= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2\\
&= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.
\end{aligned}$$ ■

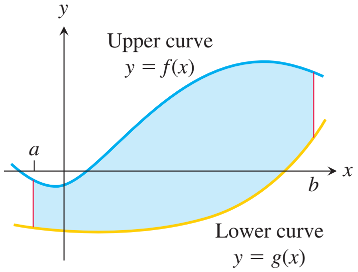


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

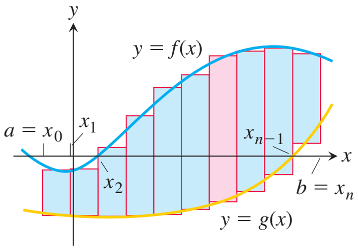


FIGURE 5.26 We approximate the region with rectangles perpendicular to the x-axis.

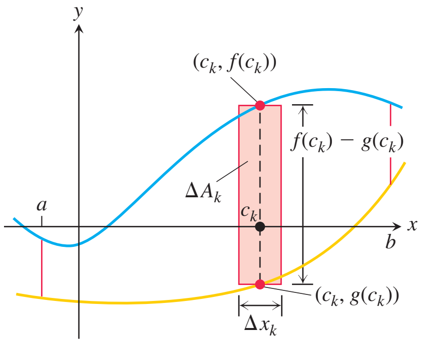


FIGURE 5.27 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.25). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area by computing an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Figure 5.26). The area of the kth rectangle (Figure 5.27) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\,\Delta x_k.$$

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann sum}$$

As $\|P\| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because f and g are continuous. The area of the region is defined to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k = \int_a^b [f(x) - g(x)]\,dx.$$

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

When applying this definition it is usually helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

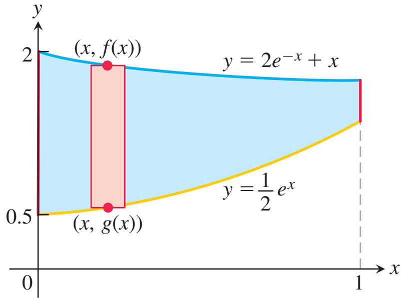


FIGURE 5.28 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

Solution Figure 5.28 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \leq x \leq 1$ is

$$\begin{aligned} A &= \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x\right] dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x\right]_0^1 \\ &= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e\right) - \left(-2 + 0 - \frac{1}{2}\right) \\ &= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051. \end{aligned}$$

■

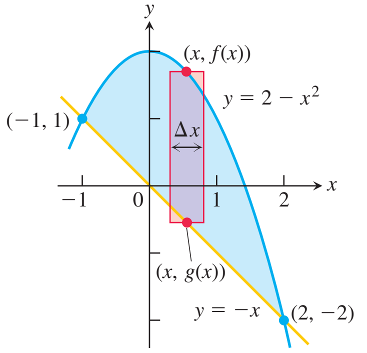


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle from a Riemann sum.

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.29). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \quad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^2 [(2 - x^2) - (-x)]\, dx \\ &= \int_{-1}^2 (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}. \end{aligned}$$

■

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

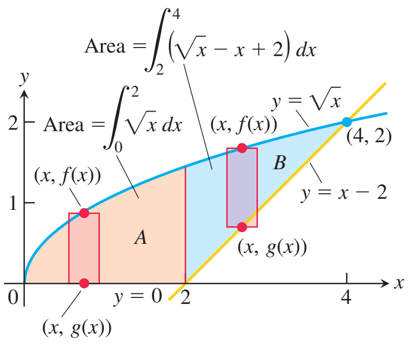


FIGURE 5.30 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.30) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.30.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$
\begin{aligned}
\sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x).\\
x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.}\\
x^2 - 5x + 4 &= 0 && \text{Rewrite.}\\
(x-1)(x-4) &= 0 && \text{Factor.}\\
x = 1, \quad x &= 4. && \text{Solve.}
\end{aligned}
$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$
\begin{aligned}
&\text{For } 0 \le x \le 2: && f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}\\
&\text{For } 2 \le x \le 4: && f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2
\end{aligned}
$$

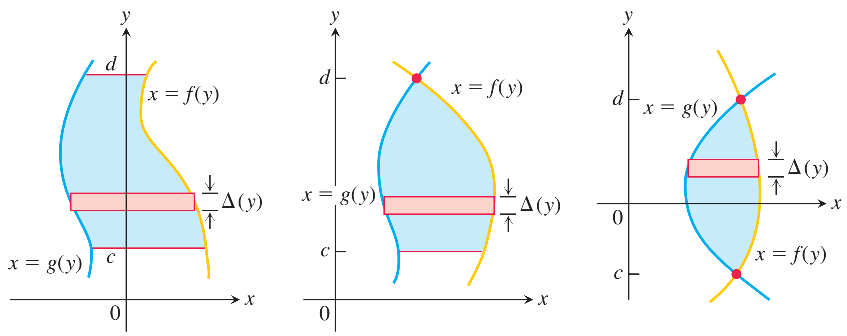
We add the areas of subregions A and B to find the total area:

$$
\begin{aligned}
\text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 \left(\sqrt{x} - x + 2\right) dx}_{\text{area of } B}\\
&= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4\\
&= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)\\
&= \frac{2}{3}(8) - 2 = \frac{10}{3}.
\end{aligned}
$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these:



use the formula

$$
A = \int_c^d [\,f(y) - g(y)\,]\,dy.
$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

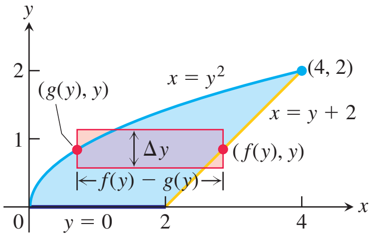


FIGURE 5.31 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 7).

EXAMPLE 7 Find the area of the region in Example 6 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.31). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \quad y &= 2 && \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$\begin{aligned} A &= \int_c^d [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy \\ &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

This is the result of Example 6, found with less work. ■

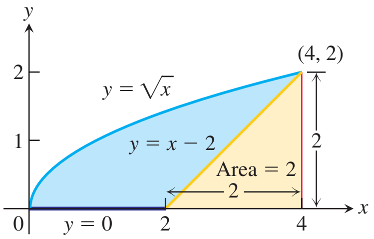


FIGURE 5.32 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

Although it was easier to find the area in Example 6 by integrating with respect to y rather than x (just as we did in Example 7), there is an easier way yet. Looking at Figure 5.32, we see that the area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, *minus* the area of an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\Big]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 7 to evaluate the integrals in Exercises 1–48.

1. a. $\int_0^3 \sqrt{y + 1}\,dy$ **b.** $\int_{-1}^0 \sqrt{y + 1}\,dy$

2. a. $\int_0^1 r\sqrt{1 - r^2}\,dr$ **b.** $\int_{-1}^1 r\sqrt{1 - r^2}\,dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\,dx$ **b.** $\int_{-\pi/4}^0 \tan x \sec^2 x\,dx$

4. **a.** $\int_0^{\pi} 3\cos^2 x \sin x\, dx$ **b.** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$

5. **a.** $\int_0^1 t^3(1+t^4)^3\, dt$ **b.** $\int_{-1}^1 t^3(1+t^4)^3\, dt$

6. **a.** $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ **b.** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$

7. **a.** $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ **b.** $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$

8. **a.** $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ **b.** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$

9. **a.** $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ **b.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

10. **a.** $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\, dx$ **b.** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\, dx$

11. **a.** $\int_0^1 t\sqrt{4+5t}\, dt$ **b.** $\int_1^9 t\sqrt{4+5t}\, dt$

12. **a.** $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\, dt$

b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\, dt$

13. **a.** $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$ **b.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

14. **a.** $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$

b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$ **16.** $\int_1^4 \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$ **18.** $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\, dt$ **20.** $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\, dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\, dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\, dy$

23. $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\, d\theta$ **24.** $\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right) dt$

25. $\int_0^{\pi/4} (1+e^{\tan\theta})\sec^2\theta\, d\theta$ **26.** $\int_{\pi/4}^{\pi/2} (1+e^{\cot\theta})\csc^2\theta\, d\theta$

27. $\int_0^{\pi} \frac{\sin t}{2-\cos t}\, dt$ **28.** $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\, d\theta$

29. $\int_1^2 \frac{2\ln x}{x}\, dx$ **30.** $\int_2^4 \frac{dx}{x\ln x}$

31. $\int_2^4 \frac{dx}{x(\ln x)^2}$ **32.** $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

33. $\int_0^{\pi/2} \tan\frac{x}{2}\, dx$ **34.** $\int_{\pi/4}^{\pi/2} \cot t\, dt$

35. $\int_0^{\pi/3} \tan^2\theta\cos\theta\, d\theta$ **36.** $\int_0^{\pi/12} 6\tan 3x\, dx$

37. $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{1+(\sin\theta)^2}$ **38.** $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x\, dx}{1+(\cot x)^2}$

39. $\int_0^{\ln\sqrt{3}} \frac{e^x\, dx}{1+e^{2x}}$ **40.** $\int_1^{e^{\pi/4}} \frac{4\, dt}{t(1+\ln^2 t)}$

41. $\int_0^1 \frac{4\, ds}{\sqrt{4-s^2}}$ **42.** $\int_0^{\sqrt[3]{2}/4} \frac{ds}{\sqrt{9-4s^2}}$

43. $\int_{\sqrt{2}}^2 \frac{\sec^2(\sec^{-1}x)\, dx}{x\sqrt{x^2-1}}$ **44.** $\int_{2/\sqrt{3}}^2 \frac{\cos(\sec^{-1}x)\, dx}{x\sqrt{x^2-1}}$

45. $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}}$ **46.** $\int_0^3 \frac{y\, dy}{\sqrt{5y+1}}$

47. $\int_0^1 \frac{\tan^{-1}x}{1+x^2}\, dx$ **48.** $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1}3x)}{1+9x^2}\, dx$

Area

Find the total areas of the shaded regions in Exercises 49–64.

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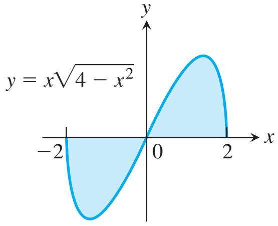

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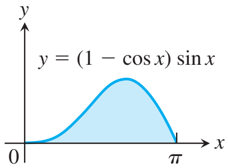

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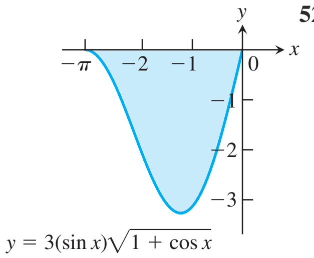

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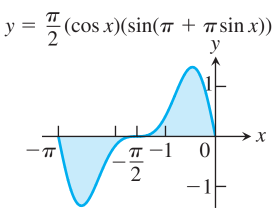

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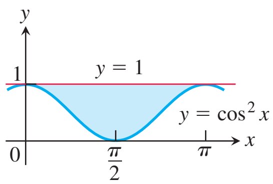

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