Hence, the truncated signal given by (4.49) can be expressed in the form
\[
\tilde{x}[n] = x[n] p\left[n - \frac{N - 1}{2}\right]
\]
(4.50)
where \(x[n]\) is the original discrete-time signal whose values are known only for \(n = 0, 1, \ldots, N - 1\).

Now let \(P(\Omega)\) denote the DTFT of the rectangular pulse \(p\left[n - \frac{N - 1}{2}\right]\). Setting \(q = (N - 1)/2\) in the result in Example 4.11 gives
\[
P(\Omega) = \frac{\sin[N\Omega/2]}{\sin(\Omega/2)} e^{-j(N-1)\Omega/2}
\]
Then, by the DTFT property involving multiplication of signals (see Table 4.2), taking the DTFT of both sides of (4.50) results in the DTFT \(\tilde{X}(\Omega)\) of the truncated signal \(\tilde{x}[n]\) given by
\[
\tilde{X}(\Omega) = X(\Omega) * P(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega - \lambda) P(\lambda) \, d\lambda
\]
(4.51)
where \(X(\Omega)\) is the DTFT of \(x[n]\). Thus, the \(N\)-point DFT \(\tilde{x}_k\) of the truncated signal \(\tilde{x}[n]\) [defined by (4.49)] is given by
\[
\tilde{x}_k = [X(\Omega) * P(\Omega)]_{\Omega = 2\pi k/N}, \quad k = 0, 1, \ldots, N - 1
\]
(4.52)
By (4.52) it is seen that the distortion in \(\tilde{x}_k\) from the desired values \(X(2\pi k/N)\) can be characterized in terms of the effect of convolving \(P(\Omega)\) with the spectrum \(X(\Omega)\) of the signal. If \(x[n]\) is not suitably small for \(n < 0\) and \(n \geq N\), in general, the sidelobes that exist in the amplitude spectrum \(|P(\Omega)|\) will result in sidelobes in the amplitude spectrum \(|X(\Omega) * P(\Omega)|\). This effect is shown in the following example:

Example 4.12 N-Point DFT

Consider the discrete-time signal \(x[n] = (0.9)^n u[n]\), \(n \geq 0\), which is plotted in Figure 4.13. From the results in Example 4.2, the DTFT is
\[
X(\Omega) = \frac{1}{1 - 0.9e^{-j\Omega}}
\]

FIGURE 4.13
Signal in Example 4.12.
The amplitude spectrum $|X(\Omega)|$ is plotted in Figure 4.14. Note that, since the signal varies rather slowly, most of the spectral content over the frequency range from 0 to $\pi$ is concentrated near the zero point $\Omega = 0$. For $N = 21$ the amplitude of the $N$-point DFT of the signal is shown in Figure 4.15. This plot was obtained by the commands

\begin{verbatim}
N = 21; n = 0:N-1;
x = 0.9.^n;
Xk = dft(x);
k = n;
stem(k,abs(Xk),'filled')
\end{verbatim}

Comparing Figures 4.14 and 4.15, we see that the amplitude of the 21-point DFT is a close approximation to the amplitude spectrum $|X(\Omega)|$. This turns out to be the case since $x[n]$ is small for $n \approx 21$ and is zero for $n < 0$.

Now consider the truncated signal $x[n]$ shown in Figure 4.16. The amplitude of the 21-point DFT of the truncated signal is plotted in Figure 4.17. This plot was generated by the following commands:

\begin{verbatim}
N = 21; n = 0:N-1;
x = 0.9.^n;
x(12:21) = zeros(1,10);
Xk = dft(x);
k = n;
stem(k,abs(Xk),'filled')
\end{verbatim}
Comparing Figures 4.17 and 4.15 reveals that the spectral content of the truncated signal has higher frequency components than those of the signal displayed in Figure 4.13. The reason for this is that the truncation at \( n = 11 \) causes an abrupt change in the signal magnitude, which introduces high-frequency components in the signal spectrum (as displayed by the DFT).

The next example shows that the sidelobes in the amplitude spectrum \( |X(\Omega)| \) can produce a phenomenon whereby spectral components can “leak” into various frequency locations as a result of the truncation process.

**Example 4.13  DFT of Truncated Sinusoid**

Suppose that the signal \( x[n] \) is the infinite-duration sinusoid \( x[n] = \cos(\Omega_0 n) \), \( -\infty < n < \infty \). From Table 4.1, we see that the DTFT of \( x[n] \) is the impulse train

\[
\sum_{i=-\infty}^{\infty} \pi[\delta(\Omega + \Omega_0 - 2\pi i) + \delta(\Omega - \Omega_0 - 2\pi i)]
\]
Chapter 4  Fourier Analysis of Discrete-Time Signals

A plot of the DTFT of \( \cos \Omega_n \) for \( -\pi \leq \Omega \leq \pi \) is shown in Figure 4.18. From the figure it is seen that, over the frequency range \( -\pi \leq \Omega \leq \pi \), all the spectral content of the signal \( \cos \Omega_n \) is concentrated at \( \Omega = \Omega_0 \) and \( \Omega = -\Omega_0 \). Now consider the truncated sinusoid \( \tilde{x}[n] = (\cos \Omega_n) \) \( n \) \( \frac{N - 1}{2} \). where \( 0 \leq \Omega_0 \leq \pi \) and \( p\left[n - \frac{N - 1}{2}\right] \) is the shifted rectangular pulse defined in Example 4.12, where \( N \) is an odd integer with \( N \geq 3 \). Then by definition of \( p\left[n - \frac{N - 1}{2}\right] \), the truncated signal is given by

\[
\tilde{x}[n] = \begin{cases} 
\cos \Omega_n, & n = 0, 1, \ldots, N - 1 \\
0, & \text{all other } n 
\end{cases}
\]

As given in Example 4.12, the DTFT \( P(\Omega) \) of the pulse \( p\left[n - \frac{N - 1}{2}\right] \) is

\[
P(\Omega) = \frac{\sin[N\Omega/2]}{\sin(\Omega/2)} e^{-j(N-1)\Omega/2}
\]

Then, by the DTFT property involving multiplication of signals, the DTFT \( \tilde{X}(\Omega) \) of the truncated signal \( \tilde{x}[n] \) is given by

\[
\tilde{X}(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\Omega - \lambda) \pi [\delta(\lambda + \Omega_0) + \delta(\lambda - \Omega_0)] d\lambda
\]

Using the shifting property of the impulse (see Section 1.1) yields

\[
\tilde{X}(\Omega) = \frac{1}{2} [P(\Omega + \Omega_0) + P(\Omega - \Omega_0)]
\]

Now the relationship (4.48) holds for the truncated signal \( \tilde{x}[n] \), and thus the \( N \)-point DFT \( \tilde{X}_k \) of \( \tilde{x}[n] \) is given by

\[
\tilde{X}_k = \tilde{X}\left(\frac{2\pi k}{N}\right) = \frac{1}{2} \left[ P\left(\frac{2\pi k}{N} + \Omega_0\right) + P\left(\frac{2\pi k}{N} - \Omega_0\right) \right], \quad k = 0, 1, \ldots, N - 1
\]

![FIGURE 4.18](image)

DTFT of \( x[n] = \cos \Omega_n \) with \( -\pi \leq \Omega \leq \pi \).
where

\[
P\left(\frac{2\pi k}{N} \pm \Omega_0\right) = \frac{\sin\left[\frac{N}{2}\left(\frac{2\pi k}{N} \pm \Omega_0\right)\right]}{\sin\left[\frac{2\pi k}{N} \pm \Omega_0\right]/2} \exp\left[-j\left(\frac{N-1}{2}\right)\left(\frac{2\pi k}{N} \pm \Omega_0\right)\right].
\]

\[k = 0, 1, 2, \ldots, N - 1\]

Suppose that \(\Omega_0 = (2\pi r)/N\) for some integer \(r\) where \(0 \leq r \leq N - 1\). This is equivalent to assuming that \(\cos \Omega_0 n\) goes through \(r\) complete periods as \(n\) is varied from \(n = 0\) to \(n = N - 1\). Then

\[
P\left(\frac{2\pi k}{N} \pm \Omega_0\right) = \frac{\sin\left(\frac{N}{2}\left(\frac{2\pi k \pm 2\pi r}{N}\right)\right)}{\sin\left(\frac{2\pi k \pm 2\pi r}{N}\right)/2} \exp\left[-jq\frac{2\pi k \pm 2\pi r}{N}\right].
\]

\[k = 0, 1, \ldots, N - 1\]

and thus

\[
P\left(\frac{2\pi k}{N} - \Omega_0\right) = \begin{cases} N, & k = r \\ 0, & k = 0, 1, \ldots, r - 1, r + 1, \ldots, N - 1 \end{cases}
\]

\[
P\left(\frac{2\pi k}{N} + \Omega_0\right) = \begin{cases} N, & k = N - r \\ 0, & k = 0, 1, \ldots, N - r - 1, N - r + 1, \ldots, N - 1 \end{cases}
\]

Finally, the DFT \(\tilde{X}_k\) is given by

\[
\tilde{X}_k = \begin{cases} N, & k = r \\ N/2, & k = N - r \\ 0, & \text{all other } k \text{ for } 0 \leq k \leq N - 1 \end{cases}
\]