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Linear Algebra with Applications

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Example 6 suggests the following result.

Theorem 2.7 **Relations and linear dependence**

The vectors $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$ are linearly dependent if (and only if) there are nontrivial relations among them.

Proof

- Suppose vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, and $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}$ is a redundant vector in this list. Then we can generate a nontrivial relation by subtracting $\vec{v}_i$ from both sides: $c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i = \vec{0}$.
- Conversely, if there is a nontrivial relation $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}$, where i is the highest index such that $c_i \neq 0$, then we can solve for $\vec{v}_i$ and thus express $\vec{v}_i$ as a linear combination of the preceding vectors:

$$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \cdots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}.$$

This shows that vector $\vec{v}_i$ is redundant, so that vectors $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, as claimed. ■

EXAMPLE 7

Suppose the column vectors of an $n \times m$ matrix A are linearly independent. Find the kernel of matrix A.

Solution

We need to solve the equation

$$A\vec{x} = \vec{0} \quad \text{or} \quad \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0} \quad \text{or} \quad x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}.$$

We see that finding the kernel of A amounts to finding the relations among the column vectors of A. By Theorem 2.7, there is only the trivial relation, with $x_1 = \cdots = x_m = 0$, so that $\ker(A) = \{\vec{0}\}$. ■

Let us summarize the findings of Example 7.

Theorem 2.8 **Kernel and relations**

The vectors in the kernel of an $n \times m$ matrix A correspond to the linear relations among the column vectors $\vec{v}_1, \dots, \vec{v}_m$ of A: The equation

$$A\vec{x} = \vec{0} \quad \text{means that} \quad x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}.$$

In particular, the column vectors of A are linearly independent if (and only if) $\ker(A) = \{\vec{0}\}$, or, equivalently, if $\operatorname{rank}(A) = m$. This condition implies that $m \leq n$.

Thus, we can find at most n linearly independent vectors in $\mathbb{R}^n$.

EXAMPLE 8 Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

to illustrate the connection between redundant column vectors, relations among the column vectors, and the kernel. See Example 6.

$$\text{Redundant column vector:} \quad \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\updownarrow$$

$$\text{Relation among column vectors:} \quad 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \vec{0}$$

$$\updownarrow$$

$$\text{Vector} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ is in } \ker(A), \text{ since } \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

■

In the following summary we list the various characterizations of linear independence discussed thus far (in Definition 2.3b, Theorem 2.7, and Theorem 2.8). We include one new characterization, (iii). The proof of the equivalence of statements (iii) and (iv) is left to the reader as Exercise 35; it is analogous to the proof of Theorem 2.7.

SUMMARY 2.9 | Various characterizations of linear independence

For a list $\vec{v}_1, \ldots, \vec{v}_m$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

i. Vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

ii. None of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ is redundant, meaning that none of them is a linear combination of preceding vectors.

iii. None of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_m$ in the list.

iv. There is only the trivial relation among the vectors $\vec{v}_1, \ldots, \vec{v}_m$, meaning that the equation $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$ has only the solution $c_1 = \cdots = c_m = 0$.

v. $\ker \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \{\vec{0}\}.$

vi. $\operatorname{rank} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = m.$

We conclude this section with an important alternative characterization of a basis. See Definition 2.3c.

EXAMPLE 9 If $\vec{v}_1, \ldots, \vec{v}_m$ is a basis of a subspace V of $\mathbb{R}^n$, and if $\vec{v}$ is a vector in V, how many solutions $c_1, \ldots, c_m$ does the equation

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

have?

Solution

There is at least one solution, since the vectors $\vec{v}_1, \ldots, \vec{v}_m$ span V (that's part of the definition of a basis). Suppose we have two representations

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ &= d_1\vec{v}_1 + \cdots + d_m\vec{v}_m. \end{aligned}$$

By subtraction, we find

$$(c_1 - d_1)\vec{v}_1 + \cdots + (c_m - d_m)\vec{v}_m = \vec{0},$$

a relation among the vectors $\vec{v}_1, \ldots, \vec{v}_m$. Since the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent, this must be the trivial relation, and we have $c_1 - d_1 = 0, \ldots, c_m - d_m = 0$, or $c_1 = d_1, \ldots, c_m = d_m$. It turns out that the two representations $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ and $\vec{v} = d_1\vec{v}_1 + \cdots + d_m\vec{v}_m$ are identical. We have shown that there is one and only one way to write $\vec{v}$ as a linear combination of the basis vectors $\vec{v}_1, \ldots, \vec{v}_m$. ■

Let us summarize.

Theorem 2.10 **Basis and unique representation**

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$.

The vectors $\vec{v}_1, \ldots, \vec{v}_m$ form a basis of V if (and only if) every vector $\vec{v}$ in V can be expressed *uniquely* as a linear combination

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

(In Section 4, we will call the coefficients $c_1, \ldots, c_m$ the *coordinates* of $\vec{v}$ with respect to the basis $\vec{v}_1, \ldots, \vec{v}_m$.)

Proof In Example 9 we have shown only one part of Theorem 2.10; we still need to verify that the uniqueness of the representation $\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$ (for every $\vec{v}$ in V) implies that $\vec{v}_1, \ldots, \vec{v}_m$ is a basis of V. Clearly, the vectors $\vec{v}_1, \ldots, \vec{v}_m$ span V, since every $\vec{v}$ in V can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

To show the linear independence of vectors $\vec{v}_1, \ldots, \vec{v}_m$, consider a relation $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$. This relation is a representation of the zero vector as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. But this representation is unique, with $c_1 = \cdots = c_m = 0$, so that $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$ must be the trivial relation. We have shown that vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent. ■

Consider the plane $V = \operatorname{im}(A) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ introduced in Example 4. (Take another look at Figure 4.)

We can write

$$\begin{aligned} \vec{v}_4 &= 1\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3 + 0\vec{v}_4 \\ &= 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 1\vec{v}_4, \end{aligned}$$

illustrating the fact that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ do not form a basis of V. However, every vector $\vec{v}$ in V can be expressed *uniquely* as a linear combination of $\vec{v}_1$ and $\vec{v}_3$ alone, meaning that the vectors $\vec{v}_1, \vec{v}_3$ do form a basis of V.

EXERCISES 2

GOAL *Check whether or not a subset of $\mathbb{R}^n$ is a subspace. Apply the concept of linear independence (in terms of Definition 2.3, Theorem 2.7, and Theorem 2.8). Apply the concept of a basis, both in terms of Definition 2.3 and in terms of Theorem 2.10.*

Which of the sets W in Exercises 1 through 3 are subspaces of $\mathbb{R}^3$?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}$

2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$

3. $W = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

4. Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$. Is span $(\vec{v}_1, \ldots, \vec{v}_m)$ necessarily a subspace of $\mathbb{R}^n$? Justify your answer.

5. Give a geometrical description of all subspaces of $\mathbb{R}^3$. Justify your answer.

6. Consider two subspaces V and W of $\mathbb{R}^n$.
   a. Is the intersection $V \cap W$ necessarily a subspace of $\mathbb{R}^n$?
   b. Is the union $V \cup W$ necessarily a subspace of $\mathbb{R}^n$?

7. Consider a nonempty subset W of $\mathbb{R}^n$ that is closed under addition and under scalar multiplication. Is W necessarily a subspace of $\mathbb{R}^n$? Explain.

8. Find a nontrivial relation among the following vectors:
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

9. Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$, with $\vec{v}_m = \vec{0}$. Are these vectors linearly independent?

In Exercises 10 through 20, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

10. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

11. $\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 11 \\ 7 \end{bmatrix}$

12. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$

15. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

17. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

18. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

20. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$

In Exercises 21 through 26, find a redundant column vector of the given matrix A, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of A. (This procedure is illustrated in Example 8.)

21. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$

Find a basis of the image of the matrices in Exercises 27 through 33.

27. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

28. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 5 & 8 \end{bmatrix}$

32. $\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

34. Consider the 5×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}.$$

We are told that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is in the kernel of A.

Write $\vec{v}_4$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

35. Show that there is a nontrivial relation among the vectors $\vec{v}_1, \dots, \vec{v}_m$ if (and only if) at least one of the vectors $\vec{v}_i$ is a linear combination of the other vectors $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_m$.

36. Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly dependent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly dependent? How can you tell?

37. Consider a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^p$ and some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Are the vectors $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_m)$ necessarily linearly independent? How can you tell?

38. **a.** Let V be a subspace of $\mathbb{R}^n$. Let m be the largest number of linearly independent vectors we can find in V. (Note that $m \le n$, by Theorem 2.8.) Choose linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in V. Show that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ span V and are therefore a basis of V. This exercise shows that any subspace of $\mathbb{R}^n$ has a basis.

If you are puzzled, think first about the special case when V is a plane in $\mathbb{R}^3$. What is m in this case?

b. Show that any subspace V of $\mathbb{R}^n$ can be represented as the image of a matrix.

39. Consider some linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$ and a vector $\vec{v}$ in $\mathbb{R}^n$ that is not contained in the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$. Are the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m, \vec{v}$ necessarily linearly independent? Justify your answer.

40. Consider an $n \times p$ matrix A and a $p \times m$ matrix B. We are told that the columns of A and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? *Hint*: Exercise 1.51 is useful.

41. Consider an $m \times n$ matrix A and an $n \times m$ matrix B (with $n \ne m$) such that $AB = I_m$. (We say that A is a *left inverse* of B.) Are the columns of B linearly independent? What about the columns of A?

42. Consider some perpendicular unit vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$. Show that these vectors are necessarily linearly independent. *Hint*: Form the dot product of $\vec{v}_i$ and both sides of the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}.$$

43. Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^n$. Are the vectors $\vec{v}_1, \vec{v}_1+\vec{v}_2, \vec{v}_1+\vec{v}_2+\vec{v}_3$ linearly independent as well? How can you tell?

44. Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ in $\mathbb{R}^n$, and let A be an invertible $m \times m$ matrix. Are the columns of the following matrix linearly independent?

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & & | \end{bmatrix} A$$

45. Are the columns of an invertible matrix linearly independent?

46. Find a basis of the kernel of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \end{bmatrix}.$$

Justify your answer carefully; that is, explain how you know that the vectors you found are linearly independent and span the kernel.

47. Consider three linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$. Find

$$\operatorname{rref}\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}.$$

48. Express the plane V in $\mathbb{R}^3$ with equation $3x_1 + 4x_2 + 5x_3 = 0$ as the kernel of a matrix A and as the image of a matrix B.

49. Express the line L in $\mathbb{R}^3$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the image of a matrix A and as the kernel of a matrix B.

50. Consider two subspaces V and W of $\mathbb{R}^n$. Let $V + W$ be the set of all vectors in $\mathbb{R}^n$ of the form $\vec{v} + \vec{w}$, where $\vec{v}$ is in V and $\vec{w}$ in W. Is $V + W$ necessarily a subspace of $\mathbb{R}^n$?

If V and W are two distinct lines in $\mathbb{R}^3$, what is $V + W$? Draw a sketch.

51. Consider two subspaces V and W of $\mathbb{R}^n$ whose intersection consists only of the vector $\vec{0}$.

a. Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in V and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$ in W. Explain why the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$ are linearly independent.

b. Consider a basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ of V and a basis $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$ of W. Explain why $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_q$ is a basis of $V + W$. See Exercise 50.

52. For which values of the constants a, b, c, d, e, and f are the following vectors linearly independent? Justify your answer.